Explosion Implosion Duality and the Laboratory Simulation of Astrophysical Systems

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Abstract

The Euler equations of ideal gas dynamics posess a remarkable nonlinear involutional symmetry which allows one to factor out an arbitrary uniform expansion or contraction of the system. The nature of this symmetry (called by cosmologists the transformation to supercomoving variables) is discussed and its origin clarified. It is pointed out that this symmetry allows one to map an explosion problem to a dual implosion problem and vice versa. The application to laboratory simulations of supernova remnants is considered; in principle this duality allows the complete three-dimensional evolution of highly structured explosion ejecta to be modelled using a static target in an implosion facility.

1 Introduction

There is much interest at present in the possible use of the new generation of high-power laser facilities (in particular the National Ignition Facility at Livermore and the Laser MegaJoule in Bordeaux) to simulate astrophysical phenomena such as supernovae. At first sight this programme appears to suffer from one obvious drawback. The phenomena one wishes to simulate generally involve *explosions* while the laser facilities are designed to produce *implosions*. Remarkably, as we will show, this is not a problem. Under certain, not too restrictive, conditions there exists an exact mathematical duality which allows one to transform an explosion problem to an implosion problem and vice versa. Thus it is possible, in a precise sense, to use *implosion* experiments to simulate *exploding* systems.

2 The duality transformation

The Euler equations of perfect gas dynamics can be conveniently written in the form

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{U} \tag{1}$$

$$\frac{D\mathbf{U}}{Dt} = -\frac{\nabla p}{\rho} \tag{2}$$

$$\frac{D\mathcal{E}}{Dt} = -(\mathcal{E}+p)\nabla\cdot\mathbf{U} \tag{3}$$

where D/Dt denotes the Lagrangian, material or convective derivative defined by

$$\frac{D}{D\,t} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \tag{4}$$

and ρ is the mass density, $\mathcal{E}$ the thermal energy density, p the pressure and $\mathbf{U}$ the velocity. In addition to these differential equations one algebraic relation is needed, an equation of state relating the pressure to the mass and energy densities the simplest being a polytropic equation of state,

$$p = (\gamma - 1)\mathcal{E} \tag{5}$$

with γ a constant. These equations are mathematically equivalent to the mass, momentum and energy conservation equations in smooth regions of the flow. Only at shocks is it necessary to revert to the fundamental conservation forms to recover the correct shock jump conditions.

Now consider the following transformation of the dependent and independent variables,

$$\begin{aligned}
\mathbf{x}^* &= a(t)^{-1}\mathbf{x}, && (6)\\
t^* &= \int a(t)^{-2} dt, && (7)\\
\rho^* &= a(t)^3 \rho, && (8)\\
p^* &= a(t)^5 p, && (9)\\
\mathbf{U}^* &= a(t)\mathbf{U} - \dot{a}(t)\mathbf{x}, && (10)\\
\mathcal{E}^* &= a(t)^5 \mathcal{E} && (11)
\end{aligned}$$

where for the moment $a(t)$ is an arbitrary function of time. Apart from the, at first sight rather strange, time-dependent scaling factors this is essentially a transformation to a coordinate system which is expanding or contracting with a scale factor $a(t)$. If we define $a^* = a^{-1}$ and note that

$$\frac{da^*}{dt^*} = a^2 \frac{d}{dt}\left(\frac{1}{a}\right) = -\frac{da}{dt} \tag{12}$$

it is easy to see that it is an involutionary transformation with inverse obtained by simply interchanging the starred and unstarred quantities

$$\begin{aligned}
\mathbf{x} &= a^*(t^*)^{-1}\mathbf{x}^*, && (13)\\
t &= \int a^*(t^*)^{-2} dt^*, && (14)\\
\rho &= a^*(t^*)^3 \rho^*, && (15)\\
p &= a^*(t^*)^5 p^*, && (16)\\
\mathbf{U} &= a^*(t^*)\mathbf{U}^* - \dot{a}^*(t^*)\mathbf{x}^*, && (17)\\
\mathcal{E} &= a^*(t^*)^5 \mathcal{E}^*. && (18)
\end{aligned}$$

Let us now consider how the dynamical equations transform under this change of variables. It is easy to see that

$$\begin{aligned}
\frac{D}{D\,t} &\to a^{*2}\frac{D}{Dt^*} && (19)\\
\nabla &\to a^* \nabla^* && (20)
\end{aligned}$$

and thus, after some elementary algebra,

$$\begin{aligned}
\frac{D\rho^*}{Dt^*} &= -\rho^*\nabla^*\cdot\mathbf{U}^*, & (21)\\
\frac{D\mathbf{U}^*}{Dt^*} &= -\frac{\nabla^*p^*}{\rho^*}+\frac{\ddot{a}^*}{a^*}\mathbf{x}^*, & (22)\\
\frac{D\mathcal{E}^*}{Dt^*} &= -(\mathcal{E}^*+p^*)\nabla^*\cdot\mathbf{U}^*+\frac{\dot{a}^*}{a^*}(3p^*-2\mathcal{E}^*). & (23)
\end{aligned}$$

Remarkably, we see that if the scale factor a is such that

$$\ddot{a}^*=\frac{d^2a^*}{dt^{*2}}=-a^2\ddot{a}=0 \tag{24}$$

and the gas is a polytrope of exponent 5/3 with $p=2\mathcal{E}/3$ then the Euler equations are *invariant* under this transformation. Note that, because the Euler equations in conservation form are algebraically equivalent to the simplified forms, the conservation forms are also invariant and thus the whole structure of ideal gas dynamics, including the Rankine-Hugoniot shock relations, is preserved.

The condition that the acceleration of the scale factor be zero, $\ddot{a}=0$, requires that $a(t)$ be a linear function of t and, without loss of generality, we can take $a=t/t_0$ where t_0 is a constant characteristic expansion time. The time transformation is then

$$t^*=\int\frac{dt}{a(t)^2}=t_0^2\int\frac{dt}{t^2}=\text{const.}-\frac{t_0^2}{t} \tag{25}$$

and it is convenient to set the constant to zero and choose

$$t^*=-\frac{t_0^2}{t},\qquad t=-\frac{t_0^2}{t^*}. \tag{26}$$

The initial singularity of the expansion in physical space occurs at $t=0$ and is mapped to $t^*=-\infty$, the long term behaviour as $t\to+\infty$ is mapped to $t^*=0$. It is important to note that in the dual representation the time variable is bounded from *above*, $t^*<0$, whereas in physical space it is bounded from *below*, $t>0$.

The remarkable result is that for an ideal gas of point particles with no internal structure (which is what the 5/3 polytrope is) hydrodynamics in a uniformly expanding system is exactly equivalent to hydrodynamics in a static system. This result, or special forms of it, appears to have been discovered a number of times by Cosmologists (where the idea of factoring out the general expansion of the universe is very natural); a recent discussion is that of Martel and Shapiro (1998) where they propose the felicitous name of "supercomoving variables" to describe this transformation. What does not seem to have been generally noted is that this transformation can be used outside the cosmological context (however Poyet and Spiegel, 1979, did use a variant in an analysis of stellar pulsations).

3 Interpretation

The fact that the transformation is exact for the gas of ideal point particles strongly hints that it is derived from a similar result for the free particle motion. In fact there is such a duality, although it is almost trivial. The freely moving point particle moves along a straight line trajectory,

$$\mathbf{x}=\mathbf{x}_0+\mathbf{v}_0t, \tag{27}$$

with starting point $\mathbf{x}_0$ and velocity $\mathbf{v}_0$. If we write this as

$$\frac{\mathbf{x}}{t} = \mathbf{v}_0 + \mathbf{x}_0 \frac{1}{t} \tag{28}$$

we see that there is a dual representation of the trajectory in which t is replaced by $1/t$, lengths are scaled by a factor proportional to time, initial points and final velocities are interchanged, but the trajectory remains a straight line. If collisions are instantaneous, localised and elastic they look the same in either system, and thus in both systems one can write down a Boltzmann equation and then derive the hydrodynamic equations as limits of moments of the Boltzmann equation. This approach also shows that higher order effects, such as viscosity and heat conduction, can formally be treated in the same way; however the resulting transformed transport coefficients will in general have unphysical time dependencies (for an application see Drury and Stewart, 1976).

This analysis also shows that similar results will hold in different numbers of spatial dimensions, but the equation of state will have to correspond to the ideal gas in that number of dimensions. In d spatial dimensions it is easy to verify that the "super-comoving" transformation takes the form

$$\begin{aligned}
\mathbf{x}^* &= a(t)^{-1}\mathbf{x}, && (29)\\
t^* &= \int a(t)^{-2} dt, && (30)\\
\rho^* &= a(t)^d \rho, && (31)\\
p^* &= a(t)^{d+2} p, && (32)\\
\mathbf{U}^* &= a(t)\mathbf{U} - \dot{a}(t)\mathbf{x}, && (33)\\
\mathcal{E}^* &= a(t)^{d+2}\mathcal{E} && (34)
\end{aligned}$$

and that the Euler equations are invariant if the gas has a polytropic equation of state such that

$$dp = 2\mathcal{E} \tag{35}$$

corresponding to an adiabatic exponent

$$\gamma = 1 + \frac{2}{d}. \tag{36}$$

An interesting way of looking at this transformation (for which we are indebted to our colleague Etienne Parizot) is that it provides an analogue in spherical geometry to the freedom that Galilei transformations allow in planar geometry. If we are looking at a planar shock, it is often convenient to transform to a reference frame where the upstream medium, or the downstream medium, or the shock itself, appears stationary. In spherical systems one cannot apply Galilei boosts because the origin is fixed, however this transformation, by allowing one to take out an arbitrary uniform expansion, gives one much the same freedom.

4 Application to a Supernova Remnant

Computational studies of the evolution of a Supernova Remnant commonly start with initial conditions of dense pressure-free ejecta expanding ballistically away from the site of the explosion, which it is convenient to locate at the coordinate origin, and interacting with a stationary, or slowly moving, ambient medium of much lower density and negligible pressure. To illustrate the application of the duality transformation let us consider the simple, if somewhat artificial, case of uniform density ejecta interacting with a uniform and stationary ambient medium in perfect spherical symmetry. Then the initial conditions correspond to

$$\rho(r,t) = \rho_0 \left(\frac{t}{t_{\rm SW}}\right)^{-3}, \tag{37}$$

$$U(r,t) = \frac{r}{t}, \tag{38}$$

$$p(r,t) = 0 \tag{39}$$

in the region $r < V_0 t$ occupied by the ejecta (V_0 is the maximum expansion speed of the ejecta) and

$$\rho(r,t) = \rho_0 \tag{40}$$

$$U(r,t) = 0, \tag{41}$$

$$p(r,t) = 0 \tag{42}$$

in the external ($r \gg V_0 t$) medium of constant density ρ_0. The sweep-up time $t_{\rm SW}$ corresponds to the point where the ejecta, if expanding unimpeded, would have a density equal to the ambient medium.

This defines the physical problem of expanding ejecta interacting with a stationary environment. Let us now consider the dual problem obtained by applying the transformation with scale factor

$$a(t) = \frac{t}{t_{\rm SW}}. \tag{43}$$

Then the dependent variables transform as

$$r^* = t_{\rm SW}\frac{r}{t}, \tag{44}$$

$$t^* = -\frac{t_{\rm SW}^2}{t} \tag{45}$$

so that the explosion, which occurs at $t = 0$ in physical problem, occurs at $t^* = -\infty$ in the dual problem. Conversely the asymptotic evolution as $t \to \infty$ in the physical problem is mapped to the behaviour at $t^* = 0$ in the dual problem.

The ejecta density in the dual problem is constant,

$$\rho^*(r^*, t^*) = a^3 \rho(r,t) = \rho_0 \tag{46}$$

and the velocity is zero, $U^* = 0$, in $r^* < V_0 t_{\rm SW}$. However the ambient medium is now time-dependent with density, in the region $r^* \gg V_0 t_{\rm SW}$

$$\rho^*(r^*, t^*) = \left(\frac{t}{t_{\rm SW}}\right)^3 \rho_0 = \left(\frac{-t^*}{t_{\rm SW}}\right)^{-3} \rho_0 \tag{47}$$

and velocity

$$U^*(r^*, t^*) = \frac{r^*}{t^*}. \tag{48}$$

Thus in the dual problem we have *stationary* ejecta interacting with an *imploding* ambient medium whereas in the physical problem we have *exploding* ejecta interacting with a *stationary* ambient medium. Instead of the initial explosion at $t = 0$ in the physical problem we have the final crunch at $t^* = 0$ in the dual problem.

The evolution in physical space of the supernova remnant structure has been often discussed and is well-known (eg Truelove and McKee, 1999; Dwarkadas and Chevalier, 1998). At early times, $t \ll t_{\rm SW}$, the bulk of the ejecta expand ballistically except for a thin interaction region on the outside consisting of a forward shock running into the ambient medium, a zone of hot shocked ambient medium, a contact discontinuity, a zone of shocked ejecta and a reverse shock propagating slowly into the ejecta. At later times, when the mass of swept up ambient material becomes comparable to the ejecta mass, the reverse shock detaches itself from the contact discontinuity and implodes on the centre and the outer forward shock approximates the self-similar Sedov solution for a strong point explosion in a cold gas.

In the dual system the interaction looks a little different, and in some ways is simpler. Initially we have the stationary sphere of high density material (which for convenience we continue to call the ejecta, although in the dual representation it has not been ejected but is simply sitting there) surrounded by a very low density converging flow. The inflowing gas has to decelerate at a shock which stands about 10% further out in radius than the edge of the ejecta. Writing for convenience $\tau = -t^*/t_{\mathrm{SW}}$ there is an exact similarity solution in which $U^* \propto \tau^{-1}$, $\rho^* \propto \tau^{-3}$ and $p^* \propto \tau^{-5}$ in the region external to the sphere of ejecta. This steeply rising pressure ($\propto \tau^{-5}$) drives the reverse shock into the ejecta and starts the implosion of the ejecta.

At later times, as the ejecta collapse, the shock in the imploding ambient medium also moves inwards thereby reducing the rate of increase of the pressure. Transforming the Sedov solution to the dual system we see that the shock radius scales as

$$r^* \propto \tau^{3/5} \tag{49}$$

and the postshock pressure as

$$p^* \propto \tau^{-19/5}. \tag{50}$$

Figure 1 attempts to show schematically the relation between the two representations. We note in passing that the dual representation is also useful for analytic and numerical studies; this aspect will be explored in a companion paper (Dwarkadas and Drury, in preparation).

5 Prospects for laboratory simulations

The perfectly symmetric explosion is neither realistic nor especially interesting; it is the easiest case to analyse numerically and there is no reason to suppose that a laboratory simulation would yield any additional information. However reality is more complicated. It is clear that the ejecta emerging from real supernova explosions are highly nonuniform on a wide range of scales and that to calculate the resulting remnant evolution in three dimensions is likely to remain a computationally challenging problem for some considerable time (cf Arnett, 1999).

The interesting implication of this work is that it should be possible with the new generation of implosion facilities to simulate precisely this problem, the interaction of highly structured ejecta with their surroundings including all the effects of spherical geometry. One can easily imagine constructing a solid target whose density distribution models the density distribution of the expanding ejecta. If this target is then used in an implosion experiment, and if the momentum loading on the surface is tailored to rise in the same manner as the pressure behind the forward shock in the dual system, a steep initial rise as τ^{-5} decreasing to $\tau^{-3.8}$, the evolution of the internal structures including all the turbulent mixing, instabilities and shock formation, should be exactly replicated.

We emphasise finally that the transformation discussed in this paper is additional to and complements the well-known linear scaling relations as excellently discussed by Ryutov et al (1999) in the astrophysical context, or Connor and Taylor (1977) in the plasma physics context. Dimensional similarity and scaling arguments are obviously central to any attempt at simulation on a laboratory scale of astrophysical systems, however precisely because they are very general and linear they cannot

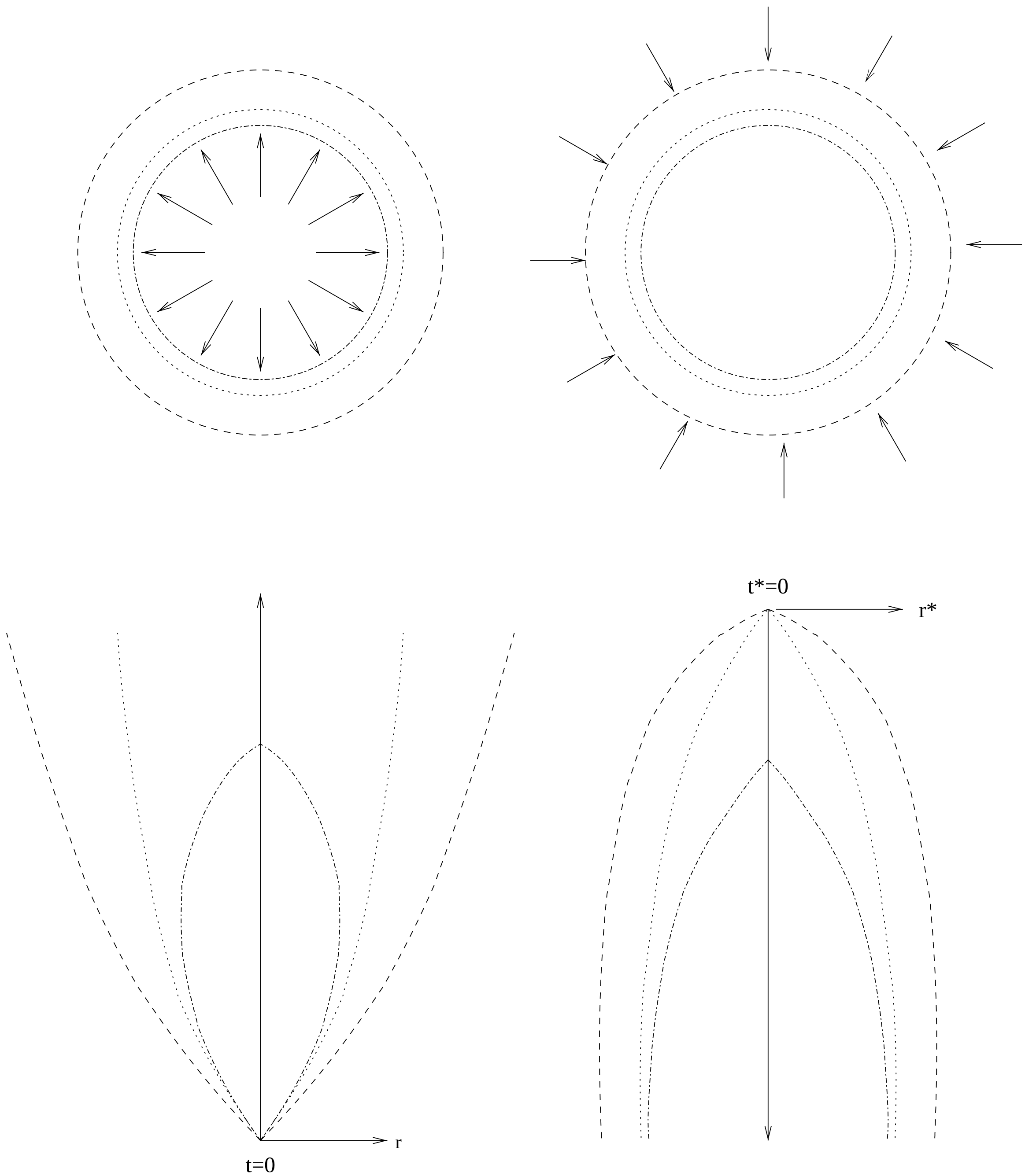

Figure 1: A schematic representation of the SNR structure and evolution as seen in physical space (left) and the dual space (right). The locations of the forward shock (dashed) , contact discontinuity (dotted) and reverse shock (dash-dotted) are indicated.

turn an explosion into an implosion. The remarkable nonlinear symmetry discussed in this paper is specific to the ideal gas equation of state, but subject to this constraint gives a powerful new degree of freedom in simulation studies by allowing an arbitrary uniform expansion or contraction to be factored out thereby transforming an explosion problem to an implosion one or vice versa.

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